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Optimum Coherent Imaging of a Limited Field of View in the Presence of Angular and Aperture Noise

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An optimal procedure is established for the reconstruction of the angular object distribution in a given field of view (FOV). The object is coherently illuminated and located in the far zone of the receiving aperture. The procedure is "uniformly" optimal in the sense of minimizing the statistical rms difference between the object distribution, modeled as a random function of the angular coordinates and its reconstructed image, for each direction belonging to the FOV. The observable complex amplitude distribution of the field on the over

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aperture is due in the general case not only to the incident field scattered by the object but also to background disturbance, or "angular noise", randomly distributed inside and outside the FOV, and is affected by "measurement noise", that is random errors introduced in measuring the aperture field. The reconstruction algorithm consists of summing a truncated series of special functions — prolate spheroidal for the linear case and their generalization for two dimensional apertures — weighted by appropriate coefficients. These coefficients depend upon the observed aperture field and upon the relative power densities associated with the object field and the various types of noise. The series is truncated to a number of terms ("effective degrees of freedom" of the image) determined through an information theoretical method: each term of the series, suitably ordered, provides an information gain less than the preceding one, and the information gain goes rapidly to zero. The relationship between information transfer and mean-squared error for each term in the image series is established. Numerical examples are discussed.

Contents

Tables

Optimum Coherent Imaging of a Limited Field of View in the Presence of Angular and Aperture Noise

I. INTRODUCTION

In recent years a substantial amount of work has been done on the problem of image reconstruction from knowledge of the field distribution on a receiving aperture. The incoherent case has been discussed by Lo,[1] who concluded that a complete restoration of a sky temperature distribution is possible in principle, despite the finite size of the aperture. The problem was reconsidered in detail by Buck and Gustincic[2] who, exploiting the properties of spheroidal functions, showed that only a limited number of functions was useful in representing the reconstructed temperature distribution. Rushforth and Harris[3] considered the effect of noise superimposed on both the object and the image distributions and took into proper account the advantages given by the a priori knowledge of the limited extension of the object in its plane. Toraldo di Francia[4] discussed the general question of the degrees of freedom of an image and pointed out the difference between the coherent

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1. Lo, Y.T. (1961) On the theoretical limitations of a radio telescope in determining the sky temperature distribution, J. Appl. Phys. 32:2052-2054.
2. Buck, G.J. and Gustincic, Jacob T. (1967) Resolution limitations of a finite aperture, IEEE Transactions on Antennas and Prop. AP-15, No. 3:376-381.
3. Rushforth, C.K. and Harris, R.W. (1968) Restoration, resolution and noise, J. Opt. Soc. Am. 58, No. 4:539-545.
4. Toraldo di Francia, G. (1969) Degrees of freedom of an image, J. Opt. Soc. Am. 59, No. 7:799-804.

and the incoherent case. Bendinelli, et al[5] established expressions for the reconstruction coefficients—in the presence of measurement noise—that minimize in a statistical sense the mean square difference between the object and its reconstruction, integrated over the extension of the object. All those studies are limited to the two-dimensional case of linear apertures.

In this paper the question of image distribution reconstruction for coherent illumination is reconsidered from first principles, without any a priori assumption of a particular processing system behind the receiving aperture, that is without postulating any particular "optical spread function." The object is assumed to be located in the aperture far zone. This being the case, the set of spherical waves constituting the scattering contributions from each point of the object can be considered locally planar at the aperture. The image reconstruction is therefore equivalent to the determination of the function of direction characterizing the complex amplitudes—referred to the aperture center—of the Plane Wave Spectrum (PWS) into which the aperture field due to the object can be decomposed. Because of its meaning the PWS associated with the object will be denoted as object angular distribution or simply object function, and mathematically modeled as a complex random function of the angular coordinates. The complex coefficients for two plane waves incident from two different directions are assumed to be statistically uncorrelated. However, their relative phase relationship is fixed in time. This is equivalent to considering the coherent case only. It is assumed that the object distribution of interest is angularly limited to a certain a priori assigned field of view (FOV). We want to reconstruct its complex values by linear processing of the complex amplitude of the field distribution observed at the receiving aperture.* The observed values are due not only to the incident field scattered from the object but also to background interference, or "angular noise", generated by scatterers from inside and outside the FOV in the aperture far zone and by "measurement noise" locally introduced at the aperture in the measurement process.*

In this paper a linear reconstruction procedure will be established by using the methods of statistical estimation. The procedure is "uniformly optimal" in the FOV in the sense of minimizing the statistical rms difference between the object distribution and its reconstructed image, for each direction of interest (ratner than

5. Bendinelli, M., Consortini, A., Ronchi, L., and Frieden, B. R. (1974) Degrees of freedom and eigenfunctions for the noisy image, J. Opt. Soc. 64, No. 11: 1498-1502.

*The situation is similar to that examined by Rushforth and Harris[3] for a somewhat different situation, since they considered the optical property of the system specified a priori through a point spread function. However, unlike the case considered in Ref. 3 in this paper all the scatterers are assumed to be in the far zone of the aperture. Consequently, the background noise in the object plane in Ref. 3 is replaced by a PWS modeled as a random function of direction and denoted as "angular noise."

the rms reconstruction error integrated over the FOV, as for example in Ref. 5). The prolate spheroidal functions, for the linear aperture, and their generalizations for two-dimensional apertures, play a fundamental role in the analysis, as they did in most of the other related work. In previous work, however, the use of spheroidal functions stemmed from their being eigunfunctions of the integral equation-defining the imaging operation—whose kernel was the optical spread function. Their relevance to reconstruction is, in a sense, far more fundamental. This was recognized by Toraldo di Francia[4] and Buck and Gustincic.[2] The appearance of spheroidal functions and their two-dimensional generalizations (discussed in this paper for relatively arbitrary geometries), is shown to be a natural consequence of the structure of the inhomogeneous integral equation defining the statistically optimal linear processing of the observed aperture field. The reconstruction algorithm consists of summing a truncated series of functions (prolate spheroidal or their generalization) weighted by coefficients depending upon the observed aperture field and upon the statistical second moments of the object distribution process and of the various types of noise. The series is truncated to a number M of terms ("Effective Degrees of Freedom") determined through considerations of information theory. Each term of the series, suitably ordered, provides an information gain less than the preceding one. The number M is such that no advantage is obtained by adding additional terms to the series for the image reconstruction. For each additional term the unconditional entropy and the entropy conditioned to the presence of a given random scene become asymptotically equal. Hence the information gain tends to zero. In terms of rms errors integrated over the FOV this fact means the following: the integrated rms error associated with a term of order $i > M$ in the reconstruction series is essentially equal to the variance of the corresponding terms of the expansion of the random scene integrated over the FOV. Therefore, those terms in the series contribute only to noise in the reconstructed image. The number M is proportional to the FOV-aperture product, unless very low and possibly unrealistic values of disturbance are present. In such cases the possibility exists of an improvement of the reconstruction accuracy beyond the limits suggested by the classical optical theory.

The domain of application of the reconstruction method here proposed is restricted to microwave frequencies. In fact, the assumption that the field can be observed at all points of the aperture implies that the field is a classical electromagnetic field, which requires a large number of quanta per degree of freedom. This condition is encountered at optical frequencies only with extremely intense fields.

2. MATHEMATICAL MODEL AND PRELIMINARY RESULTS

2.1 Object Function and Angular Noise

We will consider apertures and FOV's which are generally two-dimensional. It will be straightforward to simplify notation and results in order to deal with the simpler case of linear aperture.

Let A be a receiving aperture on the x, y plane (whose area will also be indicated by the same letter A). It is expedient for mathematical reasons to assume that the aperture has a point symmetry with respect to its center. This means, if the origin is assumed coincident with the aperture center and an aperture point is located at (x, y), that there exists another aperture point at (-x, -y). Except for this constraint of point symmetry, the geometry is arbitrary.

Let a position vector on the aperture plane be given by:

$$\underline{x} = x\hat{x} + y\hat{y}, \tag{1}$$

$\hat{x}$ and $\hat{y}$ being unit vectors in x and y directions. The rectangular coordinates in the wavenumber plane are related to the angular coordinates of a standard spherical system as follows:

$$u = \frac{2\pi}{\lambda}\sin\theta\cos\varphi, \tag{2}$$

$$v = \frac{2\pi}{\lambda}\sin\theta\sin\varphi, \tag{3}$$

where λ is the wavelength. A position vector in the u, v plane is conveniently introduced:

$$\underline{u} = u\hat{x} + v\hat{Y}. \tag{4}$$

Let $A(\underline{x})$ be a function equal to unity in the points of the aperture and zero elsewhere.

$$A(\underline{x}) = \begin{cases} 1 & \text{for } \underline{x} \in A \\ 0 & \text{for } \underline{x} \notin A \end{cases}, \tag{5}$$

the "aperture function" is defined as follows

$$\sigma(\underline{u}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A(\underline{x})e^{j\underline{u}\cdot\underline{x}}\,d^2\underline{x}, \tag{6}$$

where, of course,

$$\underline{u \cdot x} = u\,x + v\,Y\ ,$$

and $d^2\underline{x}$ is the element of the area in the x, y plane. The field of view is defined as the domain S of the u, v plane, inside which is located the object distribution to be reconstructed. S is assumed to have, like A, a point symmetry about its center in the u, v plane. The same letter S will denote the area of the FOV in the wavenumber plane. Also it proves convenient to introduce the function:

$$S(\underline{u}) = \begin{cases} 1 & \text{for } \underline{u} \in S \\ 0 & \text{for } \underline{u} \notin S \end{cases} \ . \tag{7}$$

Let the object angular distribution be a random function $g_o(\underline{u})$, different from zero only in S, which will be called the object function. The function $g_o(\underline{u})$ is the complex amplitude (referred to the aperture center) of the PWS representing the object. As mentioned in Section 1, we assume that the values of $g_o(\underline{u})$ for two different arguments are statistically uncorrelated. This physically means, intuitively speaking, that in the FOV we have no a priori information of how the object function in a certain direction $\underline{u}$ of S affects probabilistically the value of the object function in a neighboring direction. Thus, if we denote the statistical average operator by E,

$$E\left[S(\underline{u})\, g_o(\underline{u})\, S(\underline{v})\, g_o(\underline{v}) \right] = 4\pi^2\, \delta(\underline{u} - \underline{v})\, P\ \ S(\underline{u})\ , \tag{8}$$

where $\underline{v}$, like $\underline{u}$, represents an arbitrary direction and $\delta(\underline{u})$ is the two-dimensional impulse function. P_S has the physical meaning of $4\pi^2$ times the power per unit aperture area incident from an unit area of wavenumber plane.

A random function $N_1(\underline{u})$, different from zero only in the domain S of the u, v plane, is introduced in order to represent the PWS associated with background coherent disturbance, or angular noise, incident from directions belonging to the FOV. $N_1(\underline{u})$, like $g_o(\underline{u})$, is an uncorrelated random function homogeneous in S. Hence:

$$E\left[S(\underline{u})\, N_1(\underline{u})\, S(\underline{v})\, N_1(\underline{v}) \right] = 4\pi^2\, P_1\, S(\underline{u})\, \delta(\underline{u} - \underline{v})\ . \tag{9}$$

Noise is present outside the FOV. It may simply mean that the part of the scene outside the FOV is of no interest and is therefore considered a disturbance. Again such a noise $N_2(\underline{u})$ is uncorrelated:

$$E\left\{\left[1-S(\underline{u})\right]N_2(\underline{u})\left[1-S(\underline{v})\right]N_2(\underline{v})\right\} = 4\pi^2 P_2\left[1-S(\underline{u})\right]\delta(\underline{u}-\underline{v}), \quad (10)$$

and a reasonable assumption is that the statistical cross correlation between $g_o(\underline{u})$, $N_1(\underline{u})$ is zero for all values of the arguments.

In the analytical model expressed by (10) the angular noise extends throughout the whole wavenumber plane, although of course $N_2(\underline{u})$ should be zero outside the circle of the wavenumber plane corresponding to real directions, that is, outside the "visible space" G, defined as the set of points of the wavenumbers plane such that

$$|\underline{u}| \le 2\pi/\lambda .$$

The assumption (10) however is useful in simplifying the subsequent development, and has been adopted in most of the previous work in this area. In Appendix C, it will be shown that negligible error is committed by assuming the validity of (10), because in the wavenumber plane the visible space has a much greater extension than the FOV we are looking at.

2.2 Aperture Field and Aperture Noise

The received aperture field is the superposition of the plane wave spectra representing the object and the angular noise. Therefore in our scalar approximation it takes the form of a Fourier Transform:

$$f(\underline{x}) = A(\underline{x})\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{S(\underline{u})\left[g_o(\underline{u}) + N_1(\underline{u})\right] + \left[1-S(\underline{u})\right]N_2(\underline{u})\right\} e^{-j\underline{u}\cdot\underline{x}}\,d^2\underline{u}, \quad (11)$$

which defines $f(\underline{x})$ as a random function. If $\underline{\xi}$, like $\underline{x}$ represents a point of the aperture from (9 to 11), it easily follows that:

$$E\left[f(\underline{x})\,f^*(\underline{\xi})\right] = A(\underline{x})A(\underline{\xi})\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{S(\underline{u})\left(P_S + P_1\right) + \left[1-S(\underline{u})\right]P_2\right\}e^{j\underline{u}(\underline{x}-\underline{\xi})}d^2\underline{u}, \quad (12)$$

that is,

$$E\left[f(\underline{x})\,f^*(\underline{\xi})\right] = A(\underline{x})A(\underline{\xi})\left[\frac{1}{4\pi^2}\sigma_S(\underline{x}-\underline{\xi})\left(P_S + P_1 - P_2\right) + P_2\,\delta(\underline{x}-\underline{\xi})\right], \quad (13)$$

where the characteristic function of the FOV has been introduced:

$$\sigma_S(\underline{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S(\underline{u})\, e^{-j\underline{u}\cdot\underline{x}}\, d^2\underline{u}\ . \qquad (14)$$

To complete our model let us assume that additive noise $n(\underline{x})$ is present on the aperture. This means that "measurement errors" are present in the observation of the aperture field distribution. In practical cases this noise may be of a multiplicative nature. However, the additive model simplifies the mathematics and is close to reality in many cases. Thus the "observable" field distribution on the aperture is

$$o(\underline{x}) = f(\underline{x}) + n(\underline{x})\ , \qquad (15)$$

a random function having zero mean, and we assume that the measurement noise is uncorrelated:

$$E\left[A(\underline{x})\ A(\underline{\xi})\ n(\underline{x})\ n^*(\underline{\xi})\right] = \sigma_n^2\ A(\underline{x})\ \delta(\underline{x}-\underline{\xi})\ . \qquad (16)$$

This assumption is made customarily for mathematical convenience,[5] and leads to the well known difficulty of implying an infinite noise power for a continuous aperture. The difficulty is merely formal however and disappears if the rms components of aperture noise with respect to a system of orthogonal functions are considered (see Appendix C). Since the object function and the noise are uncorrelated one easily obtains by recalling (9 to 11) and (15):

$$E\left[o(\underline{x})\ o^*(\underline{\xi})\right]$$

$$= A(\underline{x})\ A(\underline{\xi})\left[\frac{1}{4\pi^2}\ \sigma_S(\underline{x}-\underline{\xi})\ \left(P_S + P_1 - P_2\right) + \left(P_2 + \sigma_n^2\right)\ \delta(\underline{x}-\underline{\xi})\right]\ . \qquad (17)$$

3. LINEAR ESTIMATION OF THE OBJECT FUNCTION

3.1 Statement of the Problem

We want to estimate the object function $g_o(\underline{u})$ by linear processing of the observable aperture field distribution $o(\underline{x})$. Hence the estimator $\hat{g}(\underline{u})$ will have the general form of linear transformation:

$$\hat{g}(\underline{u}) = S(\underline{u}) \iint_A H(\underline{u}, \underline{x})\ o(\underline{x})\ d^2\underline{x}\ . \qquad (18)$$

where $o(\underline{x})$ is given by (15) and the kernel $H(\underline{u}, \underline{x})$ is to be determined. To accomplish such a task we require that for any direction $\underline{u}$ belonging to the FOV, the statistical average of the rms estimation error be minimum:

$$e^2 = E\left[\left| \hat{g}(\underline{u}) - g_o(\underline{u}) \right|^2 \right] = \min . \tag{19}$$

As shown in Appendix A this leads to the integral equation:

$$S(\underline{u})\, A(\underline{x})\, e^{j\underline{u}\cdot\underline{x}}$$
$$= S(\underline{u})\, A(\underline{x}) \left[\left(1 + \frac{P_1 - P_2}{P_S}\right) \frac{1}{4\pi^2} \iint_A \sigma_S(\underline{x} - \underline{\xi})\, H(\underline{u}, \underline{\xi})\, d^2\underline{\xi} + \frac{P_S + \sigma_n^2}{P_S} H(\underline{u}, \underline{x}) \right] , \tag{20}$$

whose solution for $H(\underline{u}, \underline{x})$ will provide the optimum linear processing scheme.

3.2 Optimum Processor

To solve Eq. (20) consider the associate homogeneous eigenvalue equation:

$$\lambda_i \phi_i(x) = \frac{1}{4\pi^2} \iint_A \sigma_S(\underline{x} - \underline{\xi})\, \phi_i(\underline{\xi})\, d^2\underline{\xi} . \tag{21}$$

The kernel is symmetric real and positive definite, therefore the eigenvalues are positive numbers. The eigenfunctions can be chosen to be real and constitute an infinite dimensional set complete and orthogonal in A. They will be ordered for increasing values of λ_i, starting from $i = 0$. We will normalize them to unity:

$$\iint_A \phi_i(\underline{x})\, \phi_k(\underline{x})\, d^2\underline{x} = \delta_{ik} . \tag{22}$$

Let:

$$H(\underline{u}, \underline{x}) = \sum_{i=0}^{\infty} a_i(\underline{u})\, \phi_i(\underline{x}) \tag{23}$$

and insert this expression into (20), obtaining

$$S(\underline{u})\, A(\underline{x})\, e^{j\underline{u}\cdot\underline{x}}$$
$$= S(\underline{u})\, A(\underline{x}) \left[\left(1 + \frac{P_1 - P_2}{P_S}\right) \sum_{i=0}^{\infty} a_i(\underline{u})\, \lambda_i\, \phi_i(\underline{x}) + \frac{P_2 + \sigma_n^2}{P_S} \sum_{i=0}^{\infty} a_i(\underline{u})\, \phi_i(\underline{x}) \right] \tag{24}$$

To find the coefficients $a_i(\underline{u})$ we need to introduce the set of functions:

$$\Psi_i(\underline{u}) = \iint_A \psi_i(\underline{x})\, e^{j\underline{u}\cdot\underline{x}}\, d^2\underline{x}\,. \tag{25}$$

which because of Parseval's theorem form an orthogonal (although not complete) set on the wavenumber plane:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_i(\underline{u})\, \Psi_k^*(\underline{u})\, d^2\underline{u} = 4\pi^2\, \delta_{ik}\,. \tag{26}$$

A brief discussion of the properties of the set $\{\Psi_i(\underline{u})\}$ is given in Appendix B. Now multiply both the sides of (24) by $\psi_k(\underline{x})$ and integrate the aperture A. By using orthogonality and the definition (25):

$$S(\underline{u})\, \Psi_k(\underline{u}) = S(\underline{u}) \left[\left(1 + \frac{P_1 - P_2}{P_S}\right) a_k(\underline{u})\, \lambda_k + \frac{P_2 + \sigma_n^2}{P_S}\, a_k(\underline{u})\right] \tag{27}$$

and thus

$$a_k(\underline{u}) = \frac{S(\underline{u})\, \Psi_k(\underline{u})}{\left(1 + \frac{P_1 - P_2}{P_S}\right)\lambda_k + \frac{P_2 + \sigma_n^2}{P_S}}\,. \tag{28}$$

The function $H(\underline{u}, \underline{x})$ therefore is from (23):

$$H(\underline{u}, \underline{x}) \sum_{k=0}^{\infty} \frac{\Psi_k(\underline{u})\, \psi_k(\underline{x})}{\left(1 + \frac{P_1 - P_2}{P_S}\right)\lambda_k + \frac{P_2 + \sigma_n^2}{P_S}}\,. \tag{29}$$

Hence the estimator of the object function is found from (18) and (29) to be:

$$\hat{g}(\underline{u}) = S(\underline{u}) \sum_{k=0}^{\infty} \frac{\Psi_k(\underline{u})}{\left(1 + \frac{P_1 - P_2}{P_S}\right)\lambda_k + \frac{P_2 + \delta_n^2}{P_S}} \iint_A \psi_k(\underline{x})\, o(\underline{x})\, d^2\underline{x}\,. \tag{30}$$

completing the formal solution. The estimate (30) will be called the best image or simply the image of the object function $g_o(\underline{u})$.

In any practical reconstruction algorithm the series (30) is truncated after a limited number M of terms. In the next section the determination of the number M will be treated by considering the information transfer from object to image associated with each term of (30). It will be shown that unless measurement and/or angular noise have extremely low values, essentially no advantage is obtained by considering terms of order greater than N where N is defined as

$$N = \text{smallest integer} \geq AS/4\pi^2 . \tag{31}$$

Such an information theoretical approach has been preferred to the alternative approach consisting of minimizing in some sense the reconstruction error. In fact the latter method leads to certain mathematical convergence difficulties as a consequence of the assumption of a flat spectral density for $g_o(\underline{u})$, as discussed briefly in Section 5. Also the method used here provides an insight into the mechanism of information transfer from object to image.

4. INFORMATION TRANSFER FROM OBJECT TO IMAGE

4.1 Information Gain Associated with the Observable Quantities

In the processing scheme described in the previous section, the quantities

$$O_i = \iint_A \psi_k(\underline{x})\ o(\underline{x})\ d^2\underline{x} \tag{32}$$

are the "observables," and are random variables whose statistical properties are induced by these of the random function $o(\underline{x})$. Since $o(\underline{x})$ has zero mean, we have also

$$E[O_i] = 0 .$$

The second order moments of the observables (32) are found by using the orthogonal properties of the set of functions $\{\psi_i(\underline{x})\}$, and the expression (17). In this way one obtains:

$$E\left[O_i\, O_k^*\right] = \left[\lambda_i\, P_S + \lambda_i\, P_1 + \left(1 - \lambda_i\right) P_2 + \sigma_n^2\right] \delta_{ik} . \tag{33}$$

Thus the observables are uncorrelated. Also because of their definition (32), they are integrals of random functions. Thus by invoking the central limit theorem, we may assume that their probability densities are normal. The same argument can be made in regard to the conditional probability density assuming a certain object function $g_o(\underline{u})$. For the unconditional probability density the variance is from (33):

$$\sigma_i^2 = \lambda_i P_S + \lambda_i P_1 + (1 - \lambda_i) P_2 + \delta_n^2 , \tag{34}$$

and, for the conditional probability density assuming $g_o(\underline{u})$, the variance is:

$$\sigma_{ig}^2 = \lambda_i P_1 + (1 - \lambda_i) P_2 + \sigma_n^2 , \tag{35}$$

that is the variance of O_i, due only to the various types of noise.

For each observable O_i, the entropies associated with the unconditional and conditional probability densities are found under the assumption of normal distributions. Thus, for the unconditional entropy we have,[6] by using here base 2 logarithms:

$$H(i) = \log_2 (\sigma_i \sqrt{2\pi}) + \frac{1}{2 \log_e 2} \tag{36}$$

with σ_i given by (33). For the conditional entropy assuming $g_o(\underline{u})$ we have similarly:

$$H_g(i) = \log_2 (\sigma_{gi} \sqrt{2\pi}) + \frac{1}{2 \log_e 2} \tag{37}$$

with σ_{gi} given by (35). It is known that for a continuous probability density the entropy can be defined to the extent of an arbitrary additive constant. This, however, does not create any problem if one deals with a difference of two entropies. The information gain associated with the observable O_i, is in fact equal to the difference of the two entropies (36) and (37), that is,

$$I_i = H(i) - H_g(i) = \log_2 \frac{\sigma_2}{\sigma_{gi}} , \tag{38}$$

or, recalling (34-35):

$$I_i = \frac{1}{2} \log_2 \frac{\left(1 + \frac{P_1 - P_2}{P_S}\right) \lambda_i + \frac{P_2 + \sigma_n^2}{P_S}}{\frac{P_1 - P_2}{P_S} \lambda_i + \frac{P_2 + \sigma_n^2}{P_S}} . \tag{39}$$

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6. Woodward, P. M. (1953) Probability and Information Theory, with Application to Radar Pergamon Press, London, Chap. I, pp 21-25.

Since the λ_i's are a monotonic sequence decreasing with i, with a step behavior when i reaches the value N given by the aperture - FOV products (31) (Appendix B) we have:

$$\lim_{i \to \infty} I_i = 0 , \tag{40}$$

and an asymptotic expression for I_i is found to be:

$$I_i \simeq \frac{1}{2} \frac{P_S \lambda_i}{P_2 + \sigma_n^2} \log_2 e . \tag{41}$$

Equation (41) shows that the information gain is asymptotically proportional to the characteristic number λ_i associated with the ith degree of freedom multiplied by a suitably defined signal to noise ratio. Practically, unless the signal to noise ratio is very large, the effective number of degrees of freedom of the image, that is, the number M of the reconstruction terms, can be chosen equal to N in (31), because terms of higher order carry essentially no information. This conclusion is corroborated by the reconstruction error analysis in the next section, and by numerical results given in the sequel.

4.2 RMS Reconstruction Error Averaged in the FOV. Informational Compared to Statistical Approach

It is shown in Appendix B that the set of functions $\{\Psi_i(\underline{u})\}$ is orthogonal and complete in S. Thus, through a standard procedure, we can represent the object function by the expansion:

$$g_o(\underline{u}) = \sum_{i=0}^{\infty} \frac{1}{4\pi^2 \lambda_i} \Psi_i(\underline{u}) \iint_S \Psi_i^*(\underline{u})\, g_o(\underline{u})\, d^2\underline{u} . \tag{42}$$

Equation (42) is obtained without resorting to any statistical consideration. However it could have been obtained, less directly, through an application of Parseval's theorem to the integrals in Eq. (30) for the estimate, in the limiting case of absence of noise:

$$\sigma_n^2 = P_1 = P_2 = 0 . \tag{43}$$

This means that in the ideal conditions of Eq. (43) the object function can be exactly reconstructed in principle, despite the limited size of the aperture. This result is perhaps surprising, but well established, and a brief discussion of this point is given in Ref. 4. It is not of practical interest because, as soon as any random disturbance intervenes, a very different conclusion is reached. In this section we will consider the rms errors associated with each term in the estimate (30), and we will show that for terms for which the observables (32) carry zero information, the reconstruction error is equal to the *á priori* variance of the random object distribution. The result shows the equivalence of the informational and statistical approach in establishing the number of terms necessary in the reconstruction procedure (30).

Consider the rms reconstruction error averaged in the FOV, defined

$$e_a^2 = E\left\{ \iint_S \left| g_o(\underline{u}) - \hat{g}(\underline{u}) \right|^2 d^2\underline{u} \right\} . \tag{44}$$

In order to avoid complicated and after all unnecessary convergence difficulties, we assume for the moment that $g_o(\underline{u})$ is represented satisfactory by the series (42) truncated to a finite number of terms. By using (B4) of Appendix B, (30) and (42), we can write (44) as :

$$e_a^2 = \sum_i 4\pi^2 \lambda_i \, E\left[\left| \frac{\iint_A \psi_i(\underline{x})\, o(\underline{x})\, d^2\underline{x}}{\left(1 + \frac{P_1 - P_2}{P_S}\right)\lambda_i + \frac{P_2 \quad \sigma_n^2}{P_S}} - \frac{\iint_S \Psi_i(\underline{u})\, g_o(\underline{u})\, d^2\underline{u}}{4\pi^2 \lambda_i} \right|^2 \right] . \tag{45}$$

where the number of terms of the series for the time being is left unspecified. By applying Parseval's theorem to the first of the integrals in each term of the series (45) and recalling that the random object function is uncorrelated with the disturbance, one obtains through simple manipulations:

$$e_a^2 = 4\pi^2 P_S \sum_i \frac{\frac{P_1 - P_2}{P_S}\lambda_i + \frac{P_2 + \sigma_n^2}{P_S}}{\left(1 + \frac{P_1 - P_2}{P_S}\right)\lambda_i + \frac{P_2 + \sigma_n^2}{P_S}} . \tag{46}$$

By recalling (39), the expression of the rms error (46) turns out to be

$$e_a^2 = 4\pi^2 P_S \sum_i 2^{-2I_i} . \tag{47}$$

According to the discussion of Section 4.1, the information associated with the higher terms of the reconstruction series is essentially zero. For such terms Eq. (47) consistently shows that the reconstruction errors tend to a constant, proportional to the density of power per aperture unit area and per unit area of wavenumber plane associated with the object function. A further insight on the meaning of Eq. (47) is obtained by noting that the rms of any term of the series Eq. (42) for the object function integrated in the FOV is:

$$E\left[\iint_S \frac{\left|\psi_i(\underline{u})\right|^2}{\left(4\pi^2\lambda_i\right)^2} d^2u \left|\iint_S \psi_i(\underline{v})\, g_o(\underline{v})\, d^2\underline{v}\right|^2\right] = 4\pi^2 P_S , \tag{48}$$

independent of the index i. It can be established that Eq. (48) is also equal to the asymptotic value of each term of Eq. (47) when the index i tends to infinity (and I_i therefore tends to zero). This means that the integrated rms errors associated with higher order terms in the reconstruction series are equal to the á priori integrated mean squared value of the object function itself. Thus adding terms carrying zero information does not improve or deteriorate the average mean squared error of the reconstructed image. Again we reach the conclusion that, not unexpectedly, the information theoretical and statistical approaches are equivalent in expressing the circumstance that the uncertainty associated with higher order reconstruction coefficients is equal to the á priori uncertainty of the corresponding terms in the representation of the random scene. These terms are therefore useless in the reconstruction procedure.

5. PARTICULAR CASES

Two important particular situations will be now considered:

(a) Imaging a Limited Sector of a Wide-Angle Random Scene

In this case we assume that the aperture receives energy from a statistically homogeneous random scene, only a limited part of which – the one belonging to the FOV – is actually of interest. This case can be modeled by assuming that $g_o(\underline{u})$ and $N_2(\underline{u})$ are different parts – inside and outside the FOV – of the same random process, so to speak. Consequently we assume in our model that $N_1(\underline{u})$ is identically zero. Thus:

$$P_1 = 0 \;;\; P_S = P_2 . \tag{49}$$

The reconstruction procedure, Eq. (30), is in this case:

$$\hat{g}(\underline{u}) = S(\underline{u}) \frac{1}{1 + \frac{\sigma_n^2}{P_S}} \sum_k \psi_k(\underline{u}) \iint_A \psi_i(\underline{x})\, O(\underline{x})\, d^2\underline{x} . \tag{50}$$

Equation (50) shows that the reconstruction algorithm does not depend upon the ratio P_S/σ_n^2, which appears in a scale factor. In other words the form of the estimator $\hat{g}(\underline{u})$ is independent of the severity of the measurement errors and of the power received per square meter of aperture. However the measurement error affects the information transfer from the scene to the image, which for the ith reconstruction term is from Eq. (39):

$$I_i = \frac{1}{2} \log_2 \frac{1 + \frac{\sigma_n^2}{P_S}}{1 - \lambda_i + \frac{\sigma_n^2}{P_S}} . \tag{51}$$

In the next section we will see that for this case the optimal reconstruction processor is equivalent to an isoplanatic optical system.

(b) Objects in the FOV Immersed in a Wide-Angle Background of Angular Noise

Assume:

$$P_1 = P_2 = P_C . \tag{52}$$

The reconstruction algorithm is:

$$\hat{g}(\underline{u}) = S(\underline{u}) \sum_i \frac{\psi_i(\underline{u})}{\lambda_i + \frac{P_C + \sigma_n^2}{P_S}} \iint_A \psi_i(\underline{x})\, O(\underline{x})\, d^2\underline{x} , \tag{53}$$

which does depend upon the level of background noise and measurement noise with respect to the object power. The information transfer is from (39):

$$I_i = \frac{1}{2} \log_2 \frac{\lambda_i + \frac{P_C + \sigma_n^2}{P_S}}{\frac{P_C + \sigma_n^2}{P_S}} , \tag{54}$$

and goes rapidly to zero for $i > N$.

A numerical example is provided to clarify the meaning of the theoretical results obtained.

Example:

Consider the one-dimensional case of a linear aperture of length 2a. The angular extent of the FOV is $2\theta_s$, thus in the wavenumber axis its extent is:

$$2u_s = 2\frac{2\pi}{\lambda}\sin\theta_s .$$

The one dimensional equivalent of (31) is:

$$N = \text{smallest integer} \geq \frac{2a\, 2u_s}{2\pi} = \frac{2a}{\lambda}\, 2\sin\theta_s .$$

Put:

$$c = \frac{2\pi a}{\lambda}\sin\theta_s ,$$

and assume for our numerical example:

$$c = 8, \text{ that is, } N = 5 .$$

The characteristic Eq. (21) in the monodimensional case becomes the eigenvalue equation for prolate spheroidal functions, with parameter c. The first eight eigenvalues for i = 8 are listed in Table 1.

The information transfers associated with the various terms in the reconstruction series for cases (a) and (b) considered above are now numerically evaluated.

Case (a).

For various values of the ratio P_S/σ_n^2 the information contents i_i of the various terms in the image reconstruction are listed in Table 2. It is apparent that the i_i's for $i > 4$ decrease very rapidly. In fact I_5 is already negligible. Notice the weak dependence of the higher terms of the image series upon the signal to noise ratio. As expected from the theoretical considerations, five terms only carry practically the total information available.

Case (b).

In Table 3 the reconstruction coefficients for case (b) have been listed, normalized to the first (i = 0) for various values of the appropriately defined signal to noise ratio. The information transfers are listed in Table 4. Again we reach the conclusion that N = 5 terms are sufficient for the optimum reconstruction for small or moderate signal to noise ratios. However for a signal to noise ratio of the order of 30 dB or greater the possibility of a moderate increase in the accuracy of image reconstruction emerges. An inspection of Table 4 shows that I_5, I_6, and i_7 are not negligible when the signal to noise ratio is equal to 10^3.

Table 1. First Eight Eigenvalues for Prolate Spheroid at Functions, C = 8

λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
1.0000	$9.99988\cdot 10^{-1}$	$9.9700\cdot 10^{-1}$	$9.6053\cdot 10^{-1}$	$3.2028\cdot 10^{-1}$	$6.0784\cdot 10^{-2}$	$6.1263\cdot 10^{-3}$	$4.1825\cdot 10^{-4}$

Table 2. Information Transfer (N = 5), C = 8, Case (a)

P_S/σ_o^2	I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1000	4.984	4.907	3.984	2.314	$2.78\cdot 10^{-1}$	$4.5\cdot 10^{-2}$	$4.4\cdot 10^{-3}$	$3.0\cdot 10^{-4}$
100	3.329	3.321	3.141	2.176	$2.75\cdot 10^{-1}$	$4.5\cdot 10^{-2}$	$4.4\cdot 10^{-3}$	$2.9\cdot 10^{-4}$
10	1.729	1.729	1.57	1.49	$2.48\cdot 10^{-1}$	$4.1\cdot 10^{-2}$	$4.0\cdot 10^{-3}$	$2.7\cdot 10^{-4}$
2	$7.92\cdot 10^{-1}$	$7.92\cdot 10^{-1}$	$1.88\cdot 10^{-1}$	$7.37\cdot 10^{-1}$	$1.73\cdot 10^{-1}$	$3.0\cdot 10^{-2}$	$2.9\cdot 10^{-3}$	$2.6\cdot 10^{-4}$

Table 3. Reconstruction Coefficients (N = 5), Case (b), C = 8

$P_S/(P_c+\sigma_n^2)$	b_1/b_0	b_2/b_0	b_3/b_0	b_4/b_0	b_5/b_0	b_6/b_0	b_7/b_0
∞	1.0000	1.0038	1.0410	3.1222	16.452	163.23	2390.91
1000	1.0000	1.0030	1.0410	3.1156	16.202	140.46	705.79
100	1.0000	1.0030	1.0406	3.0580	14.2688	62.630	96.945
10	1.0000	1.0027	1.0371	2.6173	6.8415	10.365	10.954
2	1.0000	1.0020	1.0270	1.8286	2.6748	2.9637	2.9975

$$b_i = \frac{1}{\lambda_i + \frac{P_c + \sigma_n^2}{P_S}}$$

Table 4. Information Transfer (N = 5), C = 8, Case (b)

$P_S/(P_c+\sigma_n^2)$	I_0	I_1	I_2	I_3		I_5	I_6	I_7
1000	4.984	4.984	4.981	4.955	4.164	2.975	1.417	$2.52 \cdot 10^{-1}$
100	3.329	3.329	3.329	3.300	2.523	1.412	$3.45 \cdot 10^{-1}$	$2.96 \cdot 10^{-2}$
10	1.730	1.730	1.730	1.703	1.036	$3.43 \cdot 10^{-1}$	$4.29 \cdot 10^{-2}$	$3.01 \cdot 10^{-3}$
2	$7.92 \cdot 10^{-1}$	$7.92 \cdot 10^{-1}$	$1.92 \cdot 10^{-1}$	$7.91 \cdot 10^{-1}$	$3.57 \cdot 10^{-1}$	$4.2 \cdot 10^{-1}$	$8.78 \cdot 10^{-3}$	$6.03 \cdot 10^{-4}$

6. DISCUSSION AND FINAL REMARKS

As customary, in our treatment the scalar approximation has been assumed. For limited FOV - ten degrees or so - it is believed that no substantial error is committed. On the other hand the mathematical complexity is substantially reduced, because the analytical machinery of double orthogonal functions can be applied.

The physical reason why higher terms in the series (30) are of little or no use in the reconstruction algorithm stems from the fact that the "energy" associated with the functions $\Psi_i(\underline{u})$ is essentially confined within the FOV for $i \le N$, and outside for $i > N$ (see Appendix B). Thus higher order reconstruction terms contribute little to a faithful reproduction of the random scene inside the FOV. On the other hand by applying Parseval's theorem to the integrals in (30) it is seen that higher order terms are those maximally affected by the angular noise incident from outside the FOV. To get a better heuristic insight of the nature of the reconstruction algorithm consider the case (a) of the preceding section. In this case we singled out the sector S - the FOV - from a wide angle random scene. Truncating the appropriate estimator (50) to N terms, (N being the smallest integer greater than the aperture - FOV product), or letting the sum be extended to infinity yields informationally equivalent expressions as follows from the discussion of Section 4. We invoke now the representation for the aperture function (easy to establish):

$$\sigma_A(\underline{u} - \underline{v}) = \frac{1}{4\pi^2} \sum_{i=0}^{\infty} \Psi_i(\underline{u}) \Psi_i^*(\underline{v}) \tag{55}$$

valid for any $\underline{u}$, $\underline{v}$, and uniformly convergent for $\underline{u}$ and $\underline{v}$ both belonging to S.[7] By taking the upper limit of the sum in (50) equal to infinity, by applying Parseval's theorem to the various integrals, and interchanging the sum with the integral, one gets (neglecting a constant factor):

$$\hat{g}(\underline{u}) \simeq S(\underline{u}) \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma_A(\underline{u} - \underline{v})\, O(\underline{v})\, d^2\underline{v}\,, \tag{56}$$

where

$$O(\underline{u}) = \left[g_o(\underline{u}) + N_1(\underline{u})\right] S(\underline{u}) + N_2(\underline{u}) \left[1 - S(\underline{u})\right] + \iint_A n(\underline{x})\, e^{j\,\underline{u}\cdot\underline{x}}\, d^2\underline{x} \tag{57}$$

7. Hildebrand, Francis B. (1965) Methods of Applied Mathematics, 2nd Edition, Prentice Hall, N.J., pp 314-315.

is the angular distribution of the incident field plus an equivalent angular measurement noise—the last term in (57). Equation (56) has a simple physical interpretation. For case (a) the optimal processing is independent of the ratio P_S/σ_n^2 and is equivalent to the operation performed by a clear, aberration free (isoplanatic) optical system, with "point spread function" equal to the aperture function $\sigma_A(\underline{u})$, the observed output being limited to directions belonging to the FOV.

In different cases, like (b) in the previous section, the optimal processor depends upon the á priori knowledge of signal-to-noise ratio. However, because of the cutoff properties of λ_i, it can be inferred that, only when the noise is very small, an improved restoration is obtained by using the optimum processor instead of the optical operation (56).

To conclude these remarks we want to establish an approximate but very interesting expression for the total information that can be extracted from a limited FOV in the presence of angular and measurement noise. By invoking the cutoff property of the characteristic number λ_i, we assume that the actual values λ_i can be approximately replaced by 1 for $i \leq N$, and by 0 for $i > N$. Also assume [see (31)]:

$$N \simeq \frac{AS}{4\pi^2} .$$

Then from (39) we obtain for the total information I:

$$I \simeq \frac{AS}{4\pi^2} \frac{1}{2} \log_2 \left(1 + \frac{P_S}{P_2 + \sigma_n^2}\right) ,$$

which closely resembles the classical expression for the information transfer on a noisy channel. In the present case the product aperture FOV replaces the bandwidth-time product of Shannon's formula.

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Appendix A

The Equation for the Object Function Estimator

Let

$$H(\underline{u}, \underline{x}) = H_o(\underline{u}, \underline{x}) + \eta \, H_1(\underline{u}, \underline{x}), \tag{A 1}$$

where $H_o(\underline{u}, \underline{x})$ is the kernel satisfying (18-19), $H_1(\underline{u}, \underline{x})$ is an arbitrary function, and η is a small quantity. By recalling (19) we obtain:

$$\frac{\partial e^2}{\partial \eta} \Bigg|_{\eta = 0} = 0 , \tag{A 2}$$

because of (A 1) and the meaning of $H_o(\underline{u}, \underline{x})$. Thus from (A 2):

$$E \left\{ \left[S(\underline{u}) \, g_o(\underline{u}) - S(\underline{u}) \iint_A H_o(\underline{u}, \underline{x}) \, o(\underline{x}) \, d^2\underline{x} \right] \iint_A H_1^* (\underline{u}, \xi) \, o(\xi) \, d^2\xi \right\} . \tag{A 3}$$

Because $H_1(\underline{u}, \underline{x})$ is arbitrary, (A3) is equivalent to

$$S(\underline{u}) \, E \left[g_o(\underline{u}) \, o^*(\xi) \right] = S(\underline{u}) \iint_A H_o(\underline{u}, \underline{x}) \, E \left[o(\underline{x}) \, o^*(\xi) \right] d^2\underline{x} . \tag{A4}$$

By taking into account (16-17) we obtain:

$$A(\xi)\,\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\!\!\int_{-\infty}^{+\infty} S(\underline{u})\, E\left[g_o(\underline{u})\, g_o^*(\underline{v})\right] e^{j\underline{v}\cdot\xi}\, d^2\underline{v} = S(\underline{u})A(\xi)$$

$$\times \int_{-\infty}^{+\infty}\!\!\int_{-\infty}^{+\infty} A(\underline{x})\, H_0(\underline{u},\underline{x}) \left[\frac{1}{4\pi^2}\sigma_S(\underline{x}-\xi)\,(P_S + P_1 - P_2) + \delta(\underline{x}-\xi)\,(P_2 + \sigma_n^2\right] d^2\underline{x}\,, \tag{A5}$$

or because of (8):

$$S(\underline{u})\, A(\xi)\, P_S\, e^{j\underline{u}\cdot\underline{\xi}}$$

$$= S(\underline{u})\, A(\xi)\left[\frac{P_S + P_1 - P_2}{4\pi^2}\iint_A \sigma_S(\underline{x}-\underline{\xi})\, H_0(\underline{u},\underline{x})\, d^2\underline{x} + (P_1 + \sigma_n^2)\, H_0(\underline{u},\underline{\xi})\right] .$$

Dropping the subscript 0, not having any particular meaning, and interchanging the roles of $\underline{x}$ and $\underline{\xi}$, Eq. (20) is obtained.

Appendix B

Some Properties of the Functions $\Psi_i(\underline{u})$

By using the definition (25) and Fourier Transforming (21) we find:

$$\lambda_i \; \Psi_i(\underline{u}) = \frac{1}{4\pi^2} \iint_S \sigma_A(\underline{u} - \underline{v}) \, \Psi_i(\underline{v}) \, d^2\underline{v} \, , \tag{B1}$$

an integral equation with the same form as (21), and having identical eigenvalues. Thus the set $\{\Psi_i(\underline{u})\}$ is complete and orthogonal in S. In order to find the normalization of $\Psi_i(\underline{u})$ in S, consistent with (25), let us consider the integrated product:

$$\iint_S \Psi_i(\underline{u}) \, \Psi_k^*(\underline{u}) \, d^2\underline{u} = \iint_S d^2\underline{u} \iint_A \psi_i(\underline{x}) \, d^2\underline{x} \iint_A \psi_k(\underline{\xi}) \, e^{j\,\underline{u}(\underline{x} - \underline{\xi})} \, d^2\underline{\xi} \, . \tag{B2}$$

By interchanging the order of integrations in the left side, one obtains:

$$\iint_S \Psi_i(\underline{u}) \, \Psi_k^*(\underline{u}) \, d^2\underline{u} = \iint_A \psi_i(\underline{x}) \, d^2\underline{x} \iint_A \sigma_S(\underline{x} - \underline{\xi}) \, \psi_k(\underline{\xi}) \, d^2\underline{\xi} \, . \tag{B3}$$

By invoking (21-22) one concludes that:

$$\iint_S \Psi_i(\underline{u}) \, \Psi_k^*(\underline{u}) \, d^2\underline{u} = 4\pi^2 \lambda_i \, \delta_{ik} \, . \tag{B4}$$

Thus (25) and (B4) show that the functions $\Psi_i(\underline{u})$ form a double orthogonal system, a generalization of a known property of the prolate spheroidal functions. Also from (25) and (B4) by using Parseval's theorem:

$$\lambda_i = \frac{\iint_S \left|\Psi_i(\underline{u})\right|^2 d^2\underline{u}}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\Psi_i(\underline{u})\right|^2 d^2\underline{u}}, \tag{B5}$$

that is the characteristic value λ_i is the fraction of the total "power" associated with $\Psi_i(\underline{u})$, which belongs to S. Since the $\psi_i(\underline{x})$ are real it follows from (25) that:

$$\Psi_i(-\underline{u}) = \Psi_i^*(\underline{u}). \tag{B6}$$

The eigenvalues λ_i of (21), or (B1), are positive numbers lesser than unity. It can be shown that as a consequence of the representation (55) for the aperture function the following property holds:

$$\sum_{i=0}^{\infty} \lambda_i = \frac{AS}{4\pi^2}, \tag{B7}$$

that is the sum of the eigenvalues—trace of the integral operator in (B1)—is convergent and is proportional to the aperture - FOV product. A fundamental feature of the numbers λ_i is their cut-off property. If they are ordered with increasing values of the index, their values are close to unity for

$$\lambda_i \simeq 1, \text{ when } i \leq \text{integer part of } \frac{AS}{4\pi^2}, \tag{B8}$$

and close to zero for greater i's:

$$\lambda_i \simeq 0 \text{ for } i > \text{integer part of } \frac{AS}{4\pi^2}. \tag{B9}$$

This fundamental property has been pointed out by Slepian and Pollack* for the one-dimensional case (that is, for the prolate spheroidal functions) and has been shown to be valid for the general case by Landau.**

*Slepian, D. and Pollak, H. P. (1961) Prolate spheroidal wave functions, Fourier analysis and uncertainty — 1, Bell System Tech. J., 40:43-63.

**Landau, H. J. (1967) Necessary conditions for sampling and interpolation of certain entire functions Acta Mathematica, pp 37-52.

Appendix C

Power Associated with Each Observable

The terms (34) are the powers associated with the observables O_i's. The total power received by the aperture from the object function can be found to be by, using (B7):

$$\text{Total received object power} = P_S \frac{AS}{4\pi^2} , \tag{C1}$$

and a similar expression, with P_1 replacing P_S, holds for the total received power associated with $N_1(\underline{u})$. On the other hand the total angular noise power received from outside the FOV is:

$$\text{Total power from outside FOV} = \frac{A}{4\pi^2} \left| \pi^2 \left(\frac{2\pi}{\lambda}\right)^2 - S \right| , \tag{C2}$$

having here taken into proper account the extent of the visible space.

Assuming correctly the noise N_2 limited to visible space G, rather than extending to the entire wavenumber plane, amounts to replacing in (17) and (A5) the symbolic function $\delta(\underline{x} - \underline{\xi})$ with

$$\sigma_G(\underline{x} - \underline{\xi}) = \iint_G e^{j\underline{u}(\underline{x} - \underline{\xi})} d^2\underline{u} = \frac{4\pi}{\lambda} \frac{J_1 \left| \frac{2\pi}{\lambda} |\underline{x} - \underline{\xi}| \right|}{|\underline{x} - \underline{\xi}|} . \tag{C3}$$

We now resort to qualitative reasonings. If we consider a number of terms close to N given by (31), the functions $\psi_i(\underline{x})$ are slowly varying on the aperture (in the one dimensional case they have a number of zeros equal to their orders). On the other hand the function (C3) has a peak much sharper than that of $\sigma_S(\underline{x})$, given by (14), under the assumption of S<<G. Therefore no substantial error is committed by replacing (C3) by a delta function in integrals involving the product of (C3) with $\psi_i(\underline{x})$, if i is not substantially greater than N. In fact, more rigorously it is possible to show that, for a given product AS:

$$\lim_{\frac{S}{G} \to 0} \iint_A \psi_i(\underline{x}) \left[\frac{1}{4\pi^2} \sigma_G(\underline{x} - \underline{\xi}) - 4\pi^2 \delta(\underline{x} - \underline{\xi}) \right] d^2\underline{x} = 0 .$$

The measurement noise power for each degree of freedom is found to be

$$E \left[\iint_A \psi_i(\underline{x}) \, n(\underline{x}) \, d^2\underline{x} \iint_A \psi_i(\underline{\xi}) \, n^*(\underline{\xi}) \, d^2\xi \right] = \sigma_n^2 ,$$

because of (16) and (22). Thus σ_n^2 is equal to the rms measurement noise associated with each observable O_i, furnishing an alternative interpretation of its physical meaning. A strictly related discussion by Yaglom can be found in his book on the theory of stationary random functions.*

*Yaglom, A. M. (1965) An Introduction to the Theory of Stationary Random Functions, Prentice-Hall, N.J., pp 207-213.